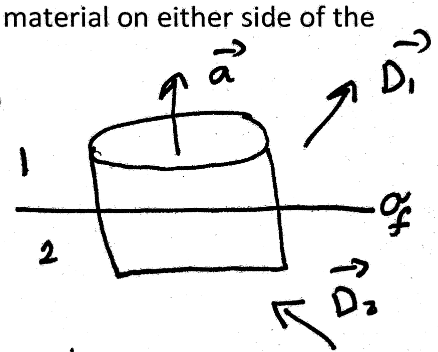
Boundary Conditions:

In general, the fields **E**, **B**, **D**, and **H** will be discontinuous at a boundary between two different media, or at a surface that carries a charge density σ or a current density **K**. The explicit form of these discontinuities can be deduced from Maxwell's equations (7.56), in their integral form

$$\left.\begin{array}{ll}\text{(i)} & \oint_{\mathcal{S}} \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} \\ \text{(ii)} & \oint_{\mathcal{S}} \mathbf{B}\cdot d\mathbf{a} = 0\end{array}\right\} \text{ over any closed surface } \mathcal{S}.$$

$$\left.\begin{array}{ll}\text{(iii)} & \oint_{\mathcal{P}} \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B}\cdot d\mathbf{a} \\ \text{(iv)} & \oint_{\mathcal{P}} \mathbf{H}\cdot d\mathbf{l} = I_{f_{\text{enc}}} + \frac{d}{dt}\int_{\mathcal{S}} \mathbf{D}\cdot d\mathbf{a}\end{array}\right\} \begin{array}{l}\text{for any surface } \mathcal{S} \\ \text{bounded by the} \\ \text{closed loop } \mathcal{P}.\end{array}$$

Applying (i) to a tiny, wafer-thin Gaussian pillbox extending just slightly into the material on either side of the boundary, we obtain:

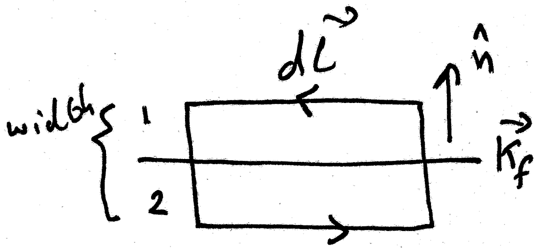


$\vec{D_1}\cdot\vec{a} - \vec{D_2}\cdot\vec{a} = \sigma_f\, a \Rightarrow D_1^{\perp} - D_2^{\perp} = \sigma_f$ --(1)

similarly from (ii), we have

$\vec{B_1}\cdot\vec{a} - \vec{B_2}\cdot\vec{a} = 0 \Rightarrow B_1^{\perp} - B_2^{\perp} = 0$ --(2)

- now applying (iii) to an Amperian loop as shown in figure, we get



$\vec{E_1}\cdot\vec{l} - \vec{E_2}\cdot\vec{l} = -\frac{d}{dt}\int_s \vec{B}\cdot d\vec{a}$

now in the limit as the width of the loop goes to zero, the flux vanishes, so

$\vec{E_1}\cdot\vec{l} - \vec{E_2}\cdot\vec{l} = 0 \Rightarrow E_1^{\parallel} - E_2^{\parallel} = 0$ -- (3)

similarly from (iv), we get

$\vec{H_1}\cdot\vec{l} - \vec{H_2}\cdot\vec{l} = I_{f_{enc}} = \vec{K_f}\cdot(\hat{n}\times\vec{l}) = (\vec{K_f}\times\hat{n})\cdot\vec{l}$; where $\hat{n}\times\vec{l}$ is normal to the Amperian loop, so

$\Rightarrow \vec{H_1}\cdot\vec{l} - \vec{H_2}\cdot\vec{l} = \vec{K_f}\times\hat{n}\cdot\vec{l}$

$\Rightarrow \vec{H_1}^{\parallel} - \vec{H_2}^{\parallel} = \vec{K_f}\times\hat{n}$ --(4)

summarizing, we have

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \quad ; \quad \text{and} \quad E_1^{\parallel} - E_2^{\parallel} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad\qquad \vec{H}_1^{\parallel} - \vec{H}_2^{\parallel} = \vec{K}_f \times \hat{n}$$

these are the general boundary conditions for electrodynamics.

- for a linear media, they can be expressed in terms of $\vec{E}$ and $\vec{B}$ alones when $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f \quad \text{and} \quad E_1^{\parallel} - E_2^{\parallel} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad\qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} - \frac{1}{\mu_2}\vec{B}_2^{\parallel} = \vec{K}_f \times \hat{n}$$

in particular, if there is no free charge ($\sigma_f = 0$) or free current ($K_f = 0$), we have

$$\varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = 0 \quad \text{and} \quad E_1^{\parallel} - E_2^{\parallel} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad\qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} - \frac{1}{\mu_2}\vec{B}_2^{\parallel} = 0$$

we will see in chapter 9, that these equations are the basis for the theory of reflection and refraction of electromagnetic waves between two media $\underline{\underline{1}}$ and $\underline{\underline{2}}$

medium 1

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medium 2